Math 6000, Fall 2020 (Prof. Kinser), Study Checks

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1. Write details of proof of the following:
α injective $\iff$ A is exact.
β surjective $\iff$ C is exact.
im $\alpha = \ker(\beta) \iff \beta$ is exact.
(In this case, $C \cong B/A$).

2. $0 \to A \xrightarrow{\alpha} B \to 0$ is exact $\iff$ α is an isomorphism.

3. (Recall definition of **triple** - pg. 61-62 on notes)
This gives a category whos objects are s.e.s.s. in R-Mod.
A morphism of sequences is an isomorphism $\iff$ α, β, η are all isomorphisms of $R-$modules $(\alpha^{-1}, \beta^{-1}, \gamma^{-1}]$: Check morphisms of inverses).

4. (Counterexample) $R = \mathbb{C}[t]$ and $M_i = \dfrac{R}{(t^i)}$ for indecomposable modules.
Check exactness.

5. Redo **Diagram Chase** for practice (pg. 65 notes) α, γ injective $\Rightarrow \beta$ injective.

5b. Try α, γ surjective $\Rightarrow \beta$ surjective (similar for isomorphism) using diagram chase.

5c. Check axioms of split sequence using diagram chase (pg. 67 notes)

6. (Prove Proposition) Let $0 \to A \to B \to C \to 0$ be a s.e.s.
Then $0 \to Hom_R(C, N) \to Hom_R(B, N) \to Hom_R(A, N)$ is exact.

7. Check that $\tilde{\sigma} : MtimesC \to \dfrac{M \otimes B}{im(id \times \alpha}$ is R-balanced. (pg.78)

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8. Check that Free Modules are Projective using relation Hom and $\oplus$.

9. Prove injective TFAE Theorem (i) $\iff$ (ii) $\iff$ (iii). [Thm. 38 in D/F]

10. (Proposition pg.92) Let $\{M_i\}_{i\in I}$ be family of $R-$modules. Then,

(i) $\oplus_{i\in I}M_i$ projective $\iff$ $M+_i$ projective.

(ii) " "" $\iff$ each flat.

(iii) $\prod_{i\in I} M_i$ injective $\iff$ each M_i injective.

(Hint: Use relations between Hom/ $\otimes$ and $\oplus/\prod$.

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10. Think about Adjoint Functors/ Functoriality (pg. 94)

11. (Abelianization) $Ab(-) : Groups \to Ab.groups$ and inclusion $Ab.groups \to Groups$.

(a) Show these are a pair of adjoint functors. (You have to figure out left vs right).

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12. Let $R = \mathbb{C}[x]$ and $M = \dfrac{R}{x^2(x-1)}$.

Find all (or some) **composition series** and compare the factors.

13. Write rigorous proof by contradiction that $\mathbb{Z}$ has no composition series.

14. $R = \mathbb{Z}$ is not Artinian.

Write this properly (pg. 109) and generalize to all PIDs.

15. (Thm) A left R module M has a composition series $\iff$ it has ACC and DCC.

16. (Qual Type Problem)

Interpret the theorem in the context of PIDs. (Overlap with theorem over PIDS).

Hints:

(i) Say take $R = \mathbb{C}[x]$ and $M = \dfrac{R}{x(x^2-1)} \oplus \dfrac{R}{x^2(x-1)^2}$ and find decomposition as in the **KS** Theorem.

(ii) Then, find composition factors of M and compare them to the composition factors of indecomposable modules in KS Theorem.

17. (**Key Theoretical Example**) Let S be a simple $R-$module. Then, $End_{R-Mod}(S)$ is a division algebra.

Prove this.

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18. Let R be a PID and $R = C[x]/I$ for some ideal I. Describe the Jacobian radical $J(R)$.

19. $Ann_R(M) = \{r \in R | rm = 0 \; \forall m \in M\}$.

(ii) $Ann_R(M)$ is a $2-$ sided ideal.

(ii) For any left ideal $I \subset R, Ann_R(R/I) \subset I$.

(iii) Show reverse containment in (ii) does NOT hold in general by computing for $R = M_2(K)$ ($K =$ field).

20. (**Good Oral Exam Questions**)

(Rotman - Proposition) There exists a surjective map of sets. Then maximal left ideals of $R \rightarrow$ simple left $R-$modules corresponds to $I \mapsto R/I$.

(pg 120-121).

21. Check $\phi : R \rightarrow^2$ defined by $\begin{bmatrix} x & a \\ y & b \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix}$ is a homomorphism of left $R-$modules.

$I_1 = \ker(\phi_1), R/I_1 \cong K^2$.

22. $R = T_n$ of upper triangular matrices.

(Check out the maximum left ideals and relate to nilpotent). - Pg. 130

23. (Ring of formal power series, $R = K[t]$ for K field).

Let $F = a_0 + a_1 + a_2 + ... = \sum_{i=0}^{\infty} a_i t^i$ with addition and multiplication as usual.

Prove that F as above is a unit $\iff a_0 \neq 0$.

(Hint: inductively construct the inverse).

24. (Theorem - Important but not deep).

There is an equivalence of categories $Rep(G) \cong F - G$ Mod.

(pg. 842-843).